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# High Order Finite Difference Schemes on Non-uniform Meshes with Good Conservation Properties

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Numerical simulation of turbulent flows (DNS or LES) requires numerical methods that are both stable and free of numerical dissipation. One way to achieve this is to enforce additional constraints, such as discrete conservation of mass, momentum, and kinetic energy. The objective of this work is to generalize the high order schemes of Morinishi *et al*. to non-uniform meshes while maintaining conservation properties of the schemes as much as possible. This generalization is achieved by preserving symmetries of the uniform mesh case. The proposed schemes do not simultaneously conserve mass, momentum, and kinetic energy. However, depending on the form of the convective term, conservation of either momentum or energy in addition to mass can be achieved. It is shown that the conservation properties of the generalized schemes are as good as those of the standard second order finite difference scheme on non-uniform meshes, while the accuracy of the new schemes is definitely superior. The predicted conservation properties are demonstrated numerically in inviscid flow simulations. © 2000 Academic Press

*Key Words:* DNS; LES; numerical method; non-uniform mesh; conservation properties.

## 1. INTRODUCTION

Numerical simulation of turbulent flows (DNS or LES) requires numerical methods that can accurately represent a wide range of spatial scales. One way to achieve a desired accuracy is to use high order finite difference schemes. However, additional constraints such as discrete conservation of mass, momentum, and kinetic energy should be enforced if one wants to ensure that unsteady flow simulations are both stable and free of numerical

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dissipation. In addition, both pressure and velocity fields must be physical. These requirements are usually achieved by using a staggered grid and enforcing continuity.

Until recently the standard second order accurate staggered grid finite difference scheme of Harlow and Welch [1] was the only scheme that simultaneously conserved mass, momentum, and kinetic energy. It was observed by Ghosal [2] that the accuracy of a second order finite difference scheme is low and fine meshes are required to achieve acceptable results. For that reason Morinishi *et al.* [3] derive the general family of fully conservative higher order accurate finite difference schemes for uniform staggered grids. Both the scheme of Harlow and Welch [1] and that of Morinishi *et al.* [3] conserve mass, momentum, and kinetic energy on a uniform mesh. However, generalizing these schemes to non-uniform meshes and preserving the conservation properties is not straightforward. For example, the generalization of the fourth order accurate finite difference scheme, suggested in [3], does not even conserve momentum. Furthermore, Morinishi *et al.* [3] mistakenly concluded that in order to construct conservative schemes, one should choose between accuracy and conservation. One of the reasons why the authors came to this conclusion may be the fact that they tried to generalize the scheme by changing the weights in the difference operators as a function of local grid spacings and preserving the order of local truncation error. As a consequence of this generalization, the resulting scheme does not preserve symmetries of the uniform mesh case. Verstappen and Veldman [4], in their analysis of the linear convective-diffusion equation on non-uniform meshes, showed that in order for the scheme to be conservative, it should preserve symmetries of the underlying operator, i.e., the convective derivative should be approximated by a skew-symmetric operator.

Another attempt to construct higher order conservative scheme was recently undertaken by Verstappen and Veldman [4, 5]. The fourth-order accuracy was achieved by using Richardson extrapolation of symmetry preserving generalization of the second-order scheme of Harlow and Welch [1]. It should be noted that the scheme of Verstappen and Veldman, when applied to uniform meshes, is different from the fourth order fully conservative schemes developed in [3] and, as shown in [3], does not conserve momentum. The major strength of the scheme proposed in [4, 5] is that it is computationally efficient and conserves both mass and energy on non-uniform meshes.

The objective of this work is to generalize the high order schemes of Morinishi *et al.* [3] to non-uniform meshes by preserving the symmetries of the uniform mesh case and to study the conservation properties of the proposed shemes.

The paper is organized as follows. Conservation properties of the mass, momentum, and kinetic energy equations for incompressible flow are reviewed in Section 2. Discrete operators, used in this paper, are defined in Section 3. The generalization of the high order schemes of Morinishi *et al.* [3] to non-uniform meshes is presented in Section 4. The conservation properties of the proposed schemes are discussed there as well. Finally, numerical tests of the conservation properties are performed in Section 5.

## 2. ANALYTICAL REQUIREMENTS

In this section, we briefly outline the analytical requirements for conservation of mass, momentum, and energy for incompressible flow. For further details we refer the reader to [3].

The continuity and momentum equations describing the motion of incompressible flow are written symbolically as

$$(Cont.) = 0, \tag{1a}$$

$$\frac{\partial u_i}{\partial t} + (Conv.)_i + (Pres.)_i + (Visc.)_i = 0, \tag{1b}$$

where

$$(Cont.) \equiv \frac{\partial u_i}{\partial x_i}, \qquad (Pres.)_i \equiv \frac{\partial p}{\partial x_i}, \qquad (Visc.)_i \equiv -\frac{\partial \tau_{ij}}{\partial x_j} \tag{2}$$

and $u_i$ is the velocity vector, $p$ is the pressure divided by density, and $\tau_{ij}$ is the viscous stress. Henceforth, $p$ will be referred to as pressure. $(Conv.)_i$ is a generic form of the convective term and will be defined below.

Conservation properties of Eqs. (1a)–(1b) will now be established. Note that Eqs. (1a)–(1b) are written in the form

$$\frac{\partial \phi}{\partial t} + {}^1Q + {}^2Q + {}^3Q + \cdots = 0. \tag{3}$$

Integrating Eq. (3) over the volume we obtain

$$\frac{\partial}{\partial t} \int\!\!\int\!\!\int_V \phi \, dV + \sum_k \int\!\!\int\!\!\int_V {}^kQ \, dV = 0.$$

We say that the term ${}^kQ$ conserves $\phi$ if the following relation holds in a periodic field:

$$\int\!\!\int\!\!\int_V {}^kQ \, dV = 0. \tag{4}$$

Following this definition of conservation, it is easy to show that Eq. (4) is satisfied automatically if ${}^kQ$ is written in divergence (conservative) form

$${}^kQ = \frac{\partial \left({}^kF_j\right)}{\partial x_j}. \tag{5}$$

Note that mass is conserved *a priori* since the continuity appears in divergence form. For the same reason the pressure $(Pres.)_i$ and viscous $(Visc.)_i$ terms conserve momentum. The convective term is also conservative *a priori* if it is written in divergence form, which is not always the case. There are four commonly used forms of the convective term. These forms are referred to as *divergence*, *advective*, *skew-symmetric*, and *rotational* forms and are defined as

$$(Div.)_i \equiv \frac{\partial u_j u_i}{\partial x_j}, \tag{6a}$$

$$(Adv.)_i \equiv u_j \frac{\partial u_i}{\partial x_j}, \tag{6b}$$

$$(Skew.)_i \equiv \frac{1}{2} \frac{\partial u_j u_i}{\partial x_j} + \frac{1}{2} u_j \frac{\partial u_i}{\partial x_j}, \tag{6c}$$

$$(Rot.)_i \equiv u_j \left( \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) + \frac{1}{2} \frac{\partial u_j u_j}{\partial x_i}. \tag{6d}$$

The four forms are connected with each other through the following analytical relations,

$$(Adv.)_i = (Div.)_i - u_i \cdot (Cont.), \tag{7a}$$

$$(Skew.)_i = \frac{1}{2}(Div.)_i + \frac{1}{2}(Adv.)_i, \tag{7b}$$

$$(Rot.)_i = (Adv.)_i, \tag{7c}$$

where $(Cont.) \equiv \partial u_i/\partial x_i$. Note that the advective, skew-symmetric, and rotational forms are conservative as long as the continuity equation is satisfied.

The transport equation of the square of a velocity component, for instance, $u_1^2/2$, can be written as

$$\frac{\partial u_1^2/2}{\partial t} + u_1 \cdot (Conv.)_1 + u_1 \cdot (Pres.)_1 + u_1 \cdot (Visc.)_1 = 0, \tag{8}$$

where $(Conv.)_i$ is a generic form of the convective term, and $(Pres.)_i$ and $(Visc.)_i$ are the pressure and viscous terms, respectively. The convective term in Eq. (8) can be written for each of the forms as

$$u_1 \cdot (Div.)_1 = \frac{\partial u_j u_1^2/2}{\partial x_j} + \frac{1}{2}u_1^2 \cdot (Cont.), \tag{9a}$$

$$u_1 \cdot (\mathrm{Adv.})_1 = \frac{\partial u_j u_1^2/2}{\partial x_j} - \frac{1}{2}u_1^2 \cdot (Cont.), \tag{9b}$$

$$u_1 \cdot (Skew.)_1 = \frac{\partial u_j u_1^2/2}{\partial x_j}. \tag{9c}$$

Note that the skew-symmetric form is conservative *a priori* in the velocity square equation. Since the rotational form is equivalent to the advective form, the four convective forms are energy conservative if the continuity equation is satisfied.

The transport equation of the kinetic energy, $K \equiv u_i u_i/2$, can be written as

$$\frac{\partial K}{\partial t} + u_i \cdot (Conv.)_i + u_i \cdot (Pres.)_i + u_i \cdot (Visc.)_i = 0. \tag{10}$$

The conservation property of the convective term can be determined in the same manner as for $u_i^2/2$. The terms involving pressure and viscous stress in Eq. (10) can be written as

$$u_i \cdot (Pres.)_i = \frac{\partial p u_i}{\partial x_i} - p \cdot (Cont.), \tag{11a}$$

$$u_i \cdot (Visc.)_i = \frac{\partial \tau_{ij} u_i}{\partial x_j} - \tau_{ij}\frac{\partial u_i}{\partial x_j}. \tag{11b}$$

The pressure term conserves kinetic energy if the continuity equation is satisfied. The viscous term is not conservative because the second term on the right-hand side of Eq. (11b) is the kinetic energy dissipation. Table I provides a summary of the conservation properties of the convective, pressure, and viscous terms in the transport equations for $u_i$, $u_1^2/2$, and $K$ for incompressible flow.

Morinishi *et al.* [3] derived a class of high order schemes for a uniform staggered grid which satisfy the conservation properties in a discrete sense. The objective of this work is

**TABLE I**
**Conservation Properties of the Convective, Pressure, and Viscous Terms in the Momentum and Transport Equations**

| | Transport equations | | |
|---|---|---|---|
| Terms in momentum eq. | $u_i$ | $u_1^2/2$ | $K$ |
| (*Div.*) | $\odot$ | $\bigcirc$ | $\bigcirc$ |
| (*Adv.*) = (*Rot.*) | $\bigcirc$ | $\bigcirc$ | $\bigcirc$ |
| (*Skew.*) | $\bigcirc$ | $\odot$ | $\odot$ |
| (*Pres.*) | $\odot$ | $\times$ | $\bigcirc$ |
| (*Visc.*) | $\odot$ | $\times$ | $\times$ |

*Notation.* $\odot$, conservative *a priori*; $\bigcirc$, conservative if (*Cont.*) = 0; $\times$, not conservative.

to generalize the higher order schemes of Morinishi *et al*. [3] to non-uniform meshes while preserving discrete conservation as much as possible.

## 3. DISCRETE OPERATORS

In order to simplify the analysis, we limit our consideration to the rectangular algebraic non-uniform meshes with non-uniform grid spacing in each of the directions $x_1$, $x_2$, and $x_3$. By algebraic grid we imply that the computational grid in physical domain is obtained by mapping a uniform computational grid in the computational domain to physical domain. Let $D = [a_1, b_1] \times [a_2, b_2] \times [a_3, b_3]$ and $\Omega = [\alpha_1, \beta_1] \times [\alpha_2, \beta_2] \times [\alpha_3, \beta_3]$ be respectively the physical and computational domains, $\mathbf{x} = (x_1, x_2, x_3)^{\mathrm{T}}$ and $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)^{\mathrm{T}}$ be coordinates in physical and computational domains, $\boldsymbol{\xi} = \mathbf{f}(\mathbf{x})$ be a nonlinear map of physical domain $D$ into computational domain, and $\Delta_1$, $\Delta_2$, $\Delta_3$ be uniform grid spacings in the respective directions in computational domain $\Omega$. In this paper we limit our consideration to the case where the mapping $\boldsymbol{\xi} = \mathbf{f}(\mathbf{x})$ can be written in the form

$$\xi_i = f_i(x_i), \qquad i = 1, \ldots, 3. \tag{12}$$

In other words, we consider only uni-directional mappings, and the computational grid in physical space can be constructed as a tensor product of one-dimensional computational grids.

Let us briefly describe the staggered grid arrangement. An example of a uniform staggered grid is shown in Fig. 1. In the case of uniform grid spacings, the choice for location of velocity and pressure points is natural: the velocity components $u_i$ ($i = 1, 2, 3$) are distributed around the pressure points. The continuity equation is centered at the pressure points while the momentum equations corresponding to each velocity component are centered at the respective velocity points. In the case of a non-uniform staggered grid, the locations of pressure and velocity points are ambiguous: these points can be determined as geometrical volume and edge centers either in physical or computational spaces. Morinishi *et al*. [3] followed the first approach. However, the generalization to non-uniform meshes suggested in [3] preserves the conservation properties only in the case of the second order scheme. The reason is that for the higher order schemes (4th order and higher) the resulting discrete operators do not preserve symmetries of the uniform mesh case. Verstappen and Veldman

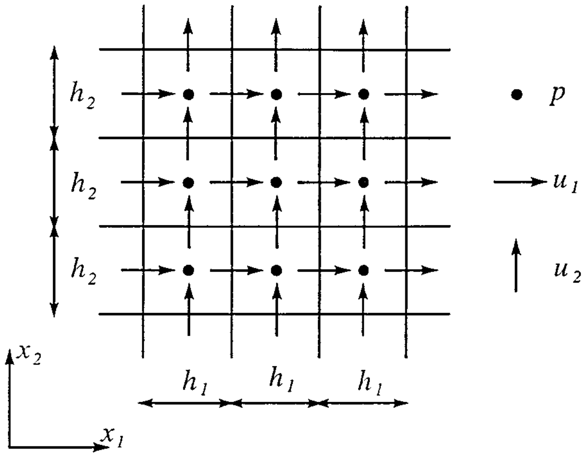


**FIG. 1.** Staggered grid arrangement.

[4] and Veldman and Rinzema [6] showed that in order for the scheme to be conservative, it should preserve symmetries of the underlying operator. The basic idea behind Veldman and Verstappen's generalization is that the differentiation operation is performed in computational space. The derivative in physical space is calculated using the local Jacobian, which can be found numerically using the same stencil and the same order accuracy as finite differencing operator in the computational space. To illustrate this idea let us consider the one-dimensional case. First, we approximate the derivative in computational space

$$\frac{\delta\phi}{\delta\xi} = \frac{\phi_{i+1} - \phi_{i-1}}{2\Delta},$$

where $\Delta$ is the uniform grid spacing. The derivative in physical space is found as

$$\frac{\delta\phi}{\delta x} = \frac{1}{J}\frac{\delta\phi}{\delta\xi}, \tag{13}$$

where $J$ is the Jacobian of the transformation $x \to \xi$, which can be found numerically by substituting $x$ for $\phi$

$$J = \frac{\delta x}{\delta\xi} = \frac{x_{i+1} - x_{i-1}}{2\Delta}.$$

This seemingly simple idea is the key which enables us to generalize the high order schemes of Morinishi *et al.* [3] to non-uniform meshes.

Let the finite difference operator in computational domain with stencil $n$ acting on $\phi$ with respect to $\xi_1$ be defined as

$$\frac{\delta_n\phi}{\delta_n\xi_1} \equiv \frac{\phi(\xi_1 + n\Delta_1/2, \xi_2, \xi_3) - \phi(\xi_1 - n\Delta_1/2, \xi_2, \xi_3)}{n\Delta_1}. \tag{14}$$

The interpolation operator with stencil $n$ acting on $\phi$ in the $\xi_1$ direction is given by

$$\overline{\phi}^{n\xi_1} \equiv \frac{\phi(\xi_1 + n\Delta_1/2, \xi_2, \xi_3) + \phi(\xi_1 - n\Delta_1/2, \xi_2, \xi_3)}{2}. \tag{15}$$

In addition, we define a special interpolation operator with stencil $n$ of the product of $\phi$ and $\psi$ in the $\xi_1$ direction,

$$\widehat{\phi\psi}^{n\xi_1} \equiv \frac{1}{2}\phi(\xi_1 + n\Delta_1/2,\ \xi_2,\ \xi_3)\psi(\xi_1 - n\Delta_1/2, \xi_2, \xi_3) + \frac{1}{2}\psi(\xi_1 + n\Delta_1/2, \xi_2, \xi_3)\phi(\xi_1 - n\Delta_1/2, \xi_2, \xi_3). \tag{16}$$

Discrete operators in the $\xi_2$ and $\xi_3$ directions are defined in the same way as for the $\xi_1$ direction.

The following identities will be needed to derive some relations later in the paper:

$$\frac{\delta_n \phi\psi^{n\xi_j}}{\delta_n \xi_j} = \phi\frac{\delta_{2n}\psi}{\delta_{2n}\xi_j} + \psi\frac{\delta_{2n}\phi}{\delta_{2n}\xi_j}, \tag{17a}$$

$$\widehat{(\phi\psi)\cdot\psi}^{n\xi_j} = \overline{\phi}^{n\xi_j}\,\widehat{\psi\psi}^{n\xi_j}, \tag{17b}$$

$$\overline{\phi}^{n\xi_j}\,\overline{\psi}^{n\xi_j} = \frac{1}{2}\overline{\phi\psi}^{n\xi_j} + \frac{1}{2}\widehat{\phi\psi}^{n\xi_j}, \tag{17c}$$

$$\frac{\delta_n\overline{\phi}^{n\xi_j}}{\delta_n\xi_j} = \frac{\delta_{2n}\phi}{\delta_{2n}\xi_j}, \tag{17d}$$

$$\frac{\delta_n\overline{\phi}^{m\xi_i}}{\delta_n\xi_j} = \frac{\overline{\delta_n\phi}^{m\xi_i}}{\delta_n\xi_j}, \tag{17e}$$

$$\overline{\psi\frac{\delta_n\phi}{\delta_n\xi_j}}^{n\xi_j} = \frac{\delta_n\psi\cdot\overline{\phi}^{n\xi_j}}{\delta_n\xi_j} - \phi\frac{\delta_n\psi}{\delta_n\xi_j}, \tag{17f}$$

$$\phi\frac{\delta_n\psi\cdot\overline{\phi}^{n\xi_j}}{\delta_n\xi_j} = \frac{1}{2}\frac{\delta_n\psi\cdot\widehat{\phi\phi}^{n\xi_j}}{\delta_n\xi_j} + \frac{1}{2}\phi\phi\frac{\delta_n\psi}{\delta_n\xi_j}. \tag{17g}$$

Note that $\xi_i$ appearing as a superscript does not follow the summation convention.

For notational convenience let us introduce the discrete finite difference operator in the physical domain,

$$\frac{\delta_n\phi}{\delta_n x_i} \equiv \frac{1}{J_i}\frac{\delta_n\phi}{\delta_n\xi_i}, \tag{18a}$$

where $J_i$ is the local Jacobian of the transformation $x_i \to \xi_i$. Note that the subscript $i$ appearing in $J_i$ in Eq. (18a) and all subsequent equations does not follow the summation convention. We emphasize that it is the form of Eq. (18a) which allows the construction of higher order schemes on non-uniform meshes with good conservation properties.

The averaging operators (15) and (16) use only functional values at grid points and do not use any information about grid spacing. Consequently, these operations are the same in physical and computational spaces:

$$\overline{\phi}^{nx_i} \equiv \overline{\phi}^{n\xi_1}, \tag{18b}$$

$$\widehat{\phi\psi}^{nx_i} \equiv \widehat{\phi\psi}^{n\xi_i}. \tag{18c}$$

We now define two concepts of discrete conservation. We say that discretized equation (3) *locally* conserves $\phi$ if all the discretized terms ${}^kQ$ can be written in the conservative form,

$$ {}^{k}Q = \sum_{n} \frac{\delta_n\left({}^{k}F_j^n\right)}{\delta_n x_j}. \tag{19} $$

This definition corresponds to the analytical divergence form of Eq. (5).

We say that discretized equation (3) is *globally* conservative if the following relation holds in a periodic field,

$$ \sum_{x_1}\sum_{x_2}\sum_{x_3} {}^{k}Q(\phi)\Delta V(\mathbf{x}) = 0, \tag{20} $$

where $\Delta V(\mathbf{x}) \equiv J\Delta V(\boldsymbol{\xi})$, $J = \prod_{k=1}^{3} J_k$ is the Jacobian of the transformation $\mathbf{x} \to \boldsymbol{\xi}$, and $\Delta V(\boldsymbol{\xi}) = \prod_{k=1}^{3} \Delta_k$ is a constant volume in the computational domain. Note that in the periodic case local conservation (19) implies global conservation. Also note that the definition (20) is a discrete analogue of Eq. (4).

## 4. FINITE DIFFERENCE SCHEMES

*4.1. Continuity and pressure terms.* We define the discrete continuity and pressure terms as

$$ (Cont. - NS2) \equiv \frac{\delta_1 u_i}{\delta_1 x_i} = 0, \tag{21} $$

$$ (Pres. - NS2)_i \equiv \frac{\delta_1 p}{\delta_1 x_i}, \tag{22} $$

where the $NS2$ denotes the second order accurate finite difference scheme on a non-uniform staggered grid. Analogously, fourth order approximations are

$$ (Cont. - NS4) \equiv \frac{9}{8}\frac{\delta_1 u_i}{\delta_1 x_i} - \frac{1}{8}\frac{\delta_3 u_i}{\delta_3 x_i} = 0, \tag{23} $$

$$ (Pres. - NS4)_i \equiv \frac{9}{8}\frac{\delta_1 p}{\delta_1 x_i} - \frac{1}{8}\frac{\delta_3 p}{\delta_3 x_i}. \tag{24} $$

Note that discrete continuity equation $(Cont. - NSn)$ is centered at the cell center (pressure) point, while pressure $(Pres. - NSn)_i$ and convective $(Conv. - NSn)_i$ terms are evaluated at the respective velocity points.

Local kinetic energy is an ambiguous quantity in a staggered grid arrangement since the individual velocity components are defined at different locations in space. Some sort of interpolation must be used in order to obtain the kinetic energy at the same point. One possible interpolation for the pressure terms in the energy equation is

$$ \frac{1}{J_i}\overline{u_i\frac{\delta_1 p}{\delta_1 \xi_i}}^{1\xi_i} = \frac{\delta_1 u_i\overline{p}^{1x_i}}{\delta_1 x_i} - p\cdot(Cont. - NS2), \tag{25} $$

$$ \frac{9}{8}\frac{1}{J_i}\overline{u_i\frac{\delta_1 p}{\delta_1 \xi_i}}^{1\xi_i} - \frac{1}{8}\frac{1}{J_i}\overline{u_i\frac{\delta_3 p}{\delta_3 \xi_i}}^{3\xi_i} = \frac{9}{8}\frac{\delta_1 u_i\overline{p}^{1x_i}}{\delta_1 x_i} - \frac{1}{8}\frac{\delta_3 u_i\overline{p}^{3x_i}}{\delta_3 x_i} - p\cdot(Cont. - NS4). \tag{26} $$

**TABLE II**
**Conservation Properties of Finite Difference Schemes for the Pressure Term on a Non-uniform Staggered Grid**

| FD schemes for momentum eq. | Transport equations $u_i$ | $K$ |
|---|---|---|
| $(Pres. - NS2)$ | $\odot$ | $\bigcirc_2$ |
| $(Pres. - NS4)$ | $\odot$ | $\bigcirc_4$ |

*Notation.* $\odot$, conservative *a priori*; $\bigcirc_2$, conservative if $(Cont. - NS2) = 0$; $\bigcirc_4$, conservative if $(Cont. - NS4) = 0$.

Therefore, pressure terms (22) and (24) conserve energy if the corresponding discrete continuity equations are satisfied. Conservation properties of the discrete pressure term on a non-uniform staggered grid are summarized in Table II.

*4.2. Second order accurate convective schemes.* As we have already mentioned, local kinetic energy $K \equiv u_i u_i/2$ cannot be defined uniquely on a staggered grid. A term is (locally) conservative in the transport equation of $K$ if the term is (locally) conservative in the transport equations of $u_1^2/2$, $u_2^2/2$, and $u_3^2/2$. Since the conservation properties of $u_2^2/2$ and $u_3^2/2$ are estimated in the same manner as for $u_1^2/2$, only the conservation properties of the convective schemes in the momentum and $u_1^2/2$ equations need to be considered.

The rotational form for a fourth and higher order convective scheme which conserves both momentum and kinetic energy on uniform mesh is not known. Therefore we limit our consideration to divergence, advective, and skew-symmetric forms. Let us define second order accurate convective schemes for non-uniform staggered grids as

$$(Div. - NS2)_i \equiv \frac{\delta_1 \overline{u_j}^{1x_i}\, \overline{u_i}^{1x_j}}{\delta_1 x_j}, \tag{27}$$

$$(Adv. - NS2)_i \equiv \frac{1}{J_j} \overline{\overline{u_j}^{1\xi_i} \frac{\delta_1 u_i}{\delta_1 \xi_j}}^{1\xi_j}, \tag{28}$$

$$(Skew. - NS2)_i \equiv \frac{1}{2}(Div. - NS2)_i + \frac{1}{2}(Adv. - NS2)_i. \tag{29}$$

Using the identities (17e), (17f), (18a), and (18b) we find that the advective $(Adv. - NS2)_i$ and divergence $(Div. - NS2)_i$ forms of the convective term are connected via

$$(Adv. - NS2)_i = (Div. - NS2)_i - u_i \frac{\delta_1 \overline{u_j}^{1x_i}}{\delta_1 x_j}. \tag{30}$$

Using (17e), Eq. (30) can be further simplified as

$$(Adv. - NS2)_i = (Div. - NS2)_i - u_i \cdot \overline{(Cont. - NS2)}^{1x_i} + u_i \cdot \left[ \overline{\frac{\delta_1 u_i}{\delta_1 x_i}}^{1x_i} - \frac{\delta_1 \overline{u_i}^{1x_i}}{\delta_1 x_i} \right], \tag{31}$$

where there is no summation over $i$. Note that the term in square brackets is the commutation error between the finite difference (18a) and averaging (18b) operators and in general is not zero, unless the grid is uniform in the $x_i$ direction.

Equations (27) and (31) are the discrete analogs of Eqs. (6a) and (7a), respectively. Clearly, Eqs. (6a) and (27) have the same structure while Eq. (31) has an additional term in it when compared to Eq. (7a). For that reason the discrete conservation properties for both advective and skew-symmetric forms of the convective term are different from analytical ones. In other words, the divergence $(Div.-NS2)_i$ form of the convective term is conservative *a priori* in the momentum equation while enforcing the discrete continuity equation is not enough to make advective $(Adv.-NS2)_i$ and skew-symmetric $(Skew.-NS2)_i$ forms conserve the momentum. This is due to the presence of a commutation error term which, in general, is non-zero for non-uniform meshes.

Using the identities (17f), (17g), and (18a)–(18c) we find that the product between $u_1$ and $(Skew.-NS2)_1$ can be rewritten as

$$u_1 \cdot (Skew.-NS2)_1 = \frac{1}{2}\frac{\delta_1 \overline{u_j}^{1x_1} \overline{\widehat{u_1 u_1}}^{1x_j}}{\delta_1 x_j}. \tag{32}$$

Therefore, $(Skew.-NS2)_1$ is conservative *a priori* in the transport equation of $u_1^2/2$. Note that in the case of the non-uniform staggered grid, the commutation error term is non-zero and neither divergence $(Div.-NS2)_i$ nor advective $(Adv.-NS2)_i$ forms of the convective term conserve kinetic energy. We also note that in the case of a uniform mesh, the commutation error is zero, and we fully recover the conservation properties described in [3]. Conservation properties of the second order accurate convective schemes on a non-uniform staggered grid are summarized in Table III. Note that the same kind of analysis for the standard generalization to a non-uniform grid of the second order scheme of Harlow and Welch [1] would lead to similar conclusions.

*4.3. Higher order accurate convective schemes.* In this section we will generalize the higher order accurate convective schemes of Morinishi *et al*. [3] for non-uniform meshes. The fourth order accurate convective schemes on a non-uniform staggered grid are defined as

$$\begin{aligned}(Div.-NS4)_i \equiv{}& \frac{9}{8}\frac{\delta_1}{\delta_1 x_j}\left\{\left(\frac{9}{8}\overline{u_j}^{1x_i} - \frac{1}{8}\overline{u_j}^{3x_i}\right)\overline{u_i}^{1x_j}\right\} \\ &- \frac{1}{8}\frac{\delta_3}{\delta_3 x_j}\left\{\left(\frac{9}{8}\overline{u_j}^{1x_i} - \frac{1}{8}\overline{u_j}^{3x_i}\right)\overline{u_i}^{3x_j}\right\},\end{aligned} \tag{33}$$

**TABLE III**
**Conservation Properties of Second Order Accurate Convective Schemes on a Non-uniform Staggered Grid**

| FD schemes for momentum eq. | Transport equations: $u_i$ | $u_1^2/2$ | $K$ |
|---|---|---|---|
| $(Div.-NS2)$ | $\odot$ | $\otimes_2$ | $\otimes_2$ |
| $(Adv.-NS2)$ | $\otimes_2$ | $\otimes_2$ | $\otimes_2$ |
| $(Skew.-NS2)$ | $\otimes_2$ | $\odot$ | $\odot$ |

*Notation.* $\odot$, conservative *a priori*; $\otimes_2$, not conservative on a non-uniform grid and conservative on a uniform grid if $(Cont.-NS2)=0$.

$$(Adv. - NS4)_i \equiv \frac{9}{8}\frac{1}{J_j}\overline{\left(\frac{9}{8}\overline{u_j}^{1\xi_i} - \frac{1}{8}\overline{u_j}^{3\xi_i}\right)\frac{\delta_1 u_i}{\delta_1 \xi_j}}^{1\xi_j}$$
$$- \frac{1}{8}\frac{1}{J_j}\overline{\left(\frac{9}{8}\overline{u_j}^{1\xi_i} - \frac{1}{8}\overline{u_j}^{3\xi_i}\right)\frac{\delta_3 u_i}{\delta_3 \xi_j}}^{3\xi_j}, \tag{34}$$

$$(Skew. - NS4)_i \equiv \frac{1}{2}(Div. - NS4)_i + \frac{1}{2}(Adv. - NS4)_i. \tag{35}$$

Using the identities (17e), (17f), (18a), and (18b) we find that the advective $(Adv. - NS4)_i$ and divergence $(Div. - NS4)_i$ forms of the convective term are connected via

$$(Adv. - NS4)_i = (Div. - NS4)_i - u_i \cdot \left(\frac{9}{8}\overline{(Cont. - NS4)}^{1x_i} - \frac{1}{8}\overline{(Cont. - NS4)}^{3x_i}\right)$$
$$+ \frac{9}{8}u_i \cdot \left(\frac{9}{8}\left[\overline{\frac{\delta_1 u_i}{\delta_1 x_i}}^{1x_i} - \frac{\delta_1 \overline{u_i}^{1x_i}}{\delta_1 x_i}\right] - \frac{1}{8}\left[\overline{\frac{\delta_3 u_i}{\delta_3 x_i}}^{1x_i} - \frac{\delta_3 \overline{u_i}^{1x_i}}{\delta_3 x_i}\right]\right)$$
$$- \frac{1}{8}u_i \cdot \left(\frac{9}{8}\left[\overline{\frac{\delta_1 u_i}{\delta_1 x_i}}^{3x_i} - \frac{\delta_1 \overline{u_i}^{3x_i}}{\delta_1 x_i}\right] - \frac{1}{8}\left[\overline{\frac{\delta_3 u_i}{\delta_3 x_i}}^{3x_i} - \frac{\delta_3 \overline{u_i}^{3x_i}}{\delta_3 x_i}\right]\right), \tag{36}$$

where there is no summation over $i$. Note that in the case of non-periodic boundary conditions the special care should be taken near the wall. This can be achieved either by decreasing the order of the scheme near the wall or by introducing *ghost points* as it is done in Morinishi *et al.* [3].

Fourth order convective schemes exhibit the same pattern as second order schemes: only the divergence form $(Div. - NS4)_i$ of the convective term is conservative *a priori* in the momentum equation. The presence of a commutation error in both advective $(Adv. - NS4)_i$ and skew-symmetric $(Skew. - NS4)_i$ forms of the convective term results in non-conservation of momentum on a non-uniform mesh.

The conservation properties for $u_1^2/2$ can be derived exactly the same way as in the previous section. Using the identities (17f), (17g), and (18a)–(18c) we obtain the relation

$$u_1 \cdot (Skew. - NS4)_1 = \frac{9}{8}\frac{\delta_1}{\delta_1 x_j}\left\{\left(\frac{9}{8}\overline{u_j}^{1x_1} - \frac{1}{8}\overline{u_j}^{3x_1}\right)\frac{\widehat{u_1 u_1}^{1x_j}}{2}\right\}$$
$$- \frac{1}{8}\frac{\delta_3}{\delta_3 x_j}\left\{\left(\frac{9}{8}\overline{u_j}^{1x_1} - \frac{1}{8}\overline{u_j}^{3x_1}\right)\frac{\widehat{u_1 u_1}^{3x_j}}{2}\right\}. \tag{37}$$

Thus, $(Skew. - NS4)_i$ is conservative *a priori* in the transport equation of $u_1^2/2$ while both the divergence $(Div. - NS4)_i$ and advective $(Adv. - NS4)_i$ forms of the convective term do not conserve kinetic energy when the staggered grid is non-uniform. In Table IV we summarize the conservation properties of the fourth order accurate convective schemes on a non-uniform staggered grid. They are identical to the properties of the second order scheme of Harlow and Welch [1].

Higher order finite difference schemes on non-uniform meshes can be constructed in the same way as for the fourth order schemes. The $n$th order accurate convective schemes on a

**TABLE IV**
**Conservation Properties of Fourth Order Accurate Convective Schemes on a Non-uniform Staggered Grid**

| | Transport equations | | |
|---|---|---|---|
| FD schemes for momentum eq. | $u_i$ | $u_1^2/2$ | $K$ |
| $(Div. - NS4)$ | $\odot$ | $\otimes_4$ | $\otimes_4$ |
| $(Adv. - NS4)$ | $\otimes_4$ | $\otimes_4$ | $\otimes_4$ |
| $(Skew. - NS4)$ | $\otimes_4$ | $\odot$ | $\odot$ |

*Notation.* $\odot$, conservative *a priori*; $\otimes_4$, not conservative on a non-uniform grid and conservative on a uniform grid if $(Cont. - NS4) = 0$.

non-uniform staggered grid are defined as

$$(Div. - NSn)_i \equiv \sum_{k=1}^{n/2} \alpha_k \frac{\delta_{(2k-1)}}{\delta_{(2k-1)} x_j} \left\{ \left( \sum_{l=1}^{n/2} \alpha_l \overline{u_j}^{(2l-1)x_i} \right) \overline{u_i}^{(2k-1)x_j} \right\}, \tag{38}$$

$$(Adv. - NSn)_i \equiv \sum_{k=1}^{n/2} \frac{\alpha_k}{J_j} \overline{\left( \sum_{l=1}^{n/2} \alpha_l \overline{u_j}^{(2l-1)\xi_i} \right) \frac{\delta_{(2k-1)} u_i}{\delta_{(2k-1)} \xi_j}}^{(2k-1)\xi_j}, \tag{39}$$

where the $\alpha_k$ are the interpolation weights. The continuity and pressure terms involve straightforward applications of the higher order interpolation operators and can be written as

$$(Cont. - NSn) \equiv \sum_{k=1}^{n/2} \alpha_k \frac{\delta_{(2k-1)} u_i}{\delta_{(2k-1)} x_i} = 0, \tag{40}$$

$$(Pres. - NSn)_i \equiv \sum_{k=1}^{n/2} \alpha_k \frac{\delta_{(2k-1)} p}{\delta_{(2k-1)} x_i}. \tag{41}$$

As an example, the sixth order accurate finite difference schemes on a staggered non-uniform grid are given by

$$(Cont. - NS6) \equiv \frac{150}{128} \frac{\delta_1 u_i}{\delta_1 x_i} - \frac{25}{128} \frac{\delta_3 u_i}{\delta_3 x_i} + \frac{3}{128} \frac{\delta_5 u_i}{\delta_5 x_i} = 0, \tag{42}$$

$$(Pres. - NS6)_i \equiv \frac{150}{128} \frac{\delta_1 p}{\delta_1 x_i} - \frac{25}{128} \frac{\delta_3 p}{\delta_3 x_i} + \frac{3}{128} \frac{\delta_5 p}{\delta_5 x_i}, \tag{43}$$

$$\begin{aligned}(Div. - NS6)_i \equiv\ & \frac{150}{128} \frac{\delta_1}{\delta_1 x_j} \left\{ \left( \frac{150}{128} \overline{u_j}^{1x_i} - \frac{25}{128} \overline{u_j}^{3x_i} + \frac{3}{128} \overline{u_j}^{5x_i} \right) \overline{u_i}^{1x_j} \right\} \\ & - \frac{25}{128} \frac{\delta_3}{\delta_3 x_j} \left\{ \left( \frac{150}{128} \overline{u_j}^{1x_i} - \frac{25}{128} \overline{u_j}^{3x_i} + \frac{3}{128} \overline{u_j}^{5x_i} \right) \overline{u_i}^{3x_j} \right\} \\ & + \frac{3}{128} \frac{\delta_5}{\delta_5 x_j} \left\{ \left( \frac{150}{128} \overline{u_j}^{1x_i} - \frac{25}{128} \overline{u_j}^{3x_i} + \frac{3}{128} \overline{u_j}^{5x_i} \right) \overline{u_i}^{5x_j} \right\},\end{aligned} \tag{44}$$

$$
\begin{aligned}
(Adv.-NS6)_i \equiv\ & \frac{150}{128}\frac{1}{J_j}\overline{\left(\frac{150}{128}\overline{u_j}^{1\xi_i}-\frac{25}{128}\overline{u_j}^{3\xi_i}+\frac{3}{128}\overline{u_j}^{5\xi_i}\right)\frac{\delta_1 u_i}{\delta_1\xi_j}}^{1\xi_j}\\
&-\frac{25}{128}\frac{1}{J_j}\overline{\left(\frac{150}{128}\overline{u_j}^{1\xi_i}-\frac{25}{128}\overline{u_j}^{3\xi_i}+\frac{3}{128}\overline{u_j}^{5\xi_i}\right)\frac{\delta_3 u_i}{\delta_3\xi_j}}^{3\xi_j}\\
&+\frac{3}{128}\frac{1}{J_j}\overline{\left(\frac{150}{128}\overline{u_j}^{1\xi_i}-\frac{25}{128}\overline{u_j}^{3\xi_i}+\frac{3}{128}\overline{u_j}^{5\xi_i}\right)\frac{\delta_5 u_i}{\delta_5\xi_j}}^{5\xi_j},
\end{aligned}
\tag{45}
$$

$$
(Skew.-NS6)_i \equiv \frac{1}{2}(Div.-NS6)_i+\frac{1}{2}(Adv.-NS6)_i. \tag{46}
$$

## 5. PERIODIC INVISCID FLOW SIMULATIONS

To confirm the results of the previous sections numerically, three-dimensional inviscid channel flow simulations are performed. The flow field is assumed to be periodic in the streamwise $x_1$ and spanwise $x_3$ directions. The fourth order accurate finite difference scheme is used for the convective term. The zero-normal velocity boundary conditions are assumed along the walls. Solenoidal initial velocity fields are generated using homogeneous random numbers. A third order Runge–Kutta (RK3) method of Spalart *et al.* [7] is used for time integration. The splitting method by Dukowicz and Dvinsky [8] is used to enforce the solenoidal condition. The resulting discrete Poisson's equation for the pressure is solved using a discrete Fourier transform in the periodic directions and a penta-diagonal direct matrix solver in the wall normal direction. The computational box is $2\pi\times 2\times 2\pi$ and $16\times 16\times 16$ mesh points are used. The grid spacings in the periodic directions are uniform. The wall normal grid is stretched using a hyperbolic-tangent function

$$
x_2(j)=\frac{\tanh(\gamma(2j/N_2-1))}{\tanh(\gamma)},\qquad j=0,\ldots,N_2. \tag{47}
$$

Numerical tests are performed for $\gamma=3$.

The analytical conservation requirements dictate that the total momentum, $\langle u_i\rangle$, and total kinetic energy, $\langle K\rangle\equiv\frac{1}{2}\langle u_1^2+u_2^2+u_3^2\rangle$, should be conserved in time. We normalize the initial velocity field in such a way that $\langle u_1|_{t=0}\rangle=\langle u_3|_{t=0}\rangle=0$ and $\langle K|_{t=0}\rangle=1$. Due to the fact that grid spacing is uniform in the streamwise and spanwise directions, the convective schemes have much better conservation properties. Since commutation error in Eq. (36) is zero for $i=1,3$, both advective and skew-symmetric forms of the convective term conserve momentum in $x_1$ and $x_3$ directions. However, the commutation error between averaging and differencing operators in the wall normal direction is not zero. Consequently, the kinetic energy is still conserved only for the skew-symmetric form of the convective term.

The conservation of momentum is confirmed numerically up to machine accuracy. Surprisingly, the momentum is conserved for all three forms of the convective term in all three directions even though the grid in wall normal direction is not uniform. We attribute this to the specific properties of the inviscid flow between parallel plates.

As we have already mentioned, the total kinetic energy is also an ambiguous quantity since it cannot be defined uniquely on a staggered grid. In this paper we used the following

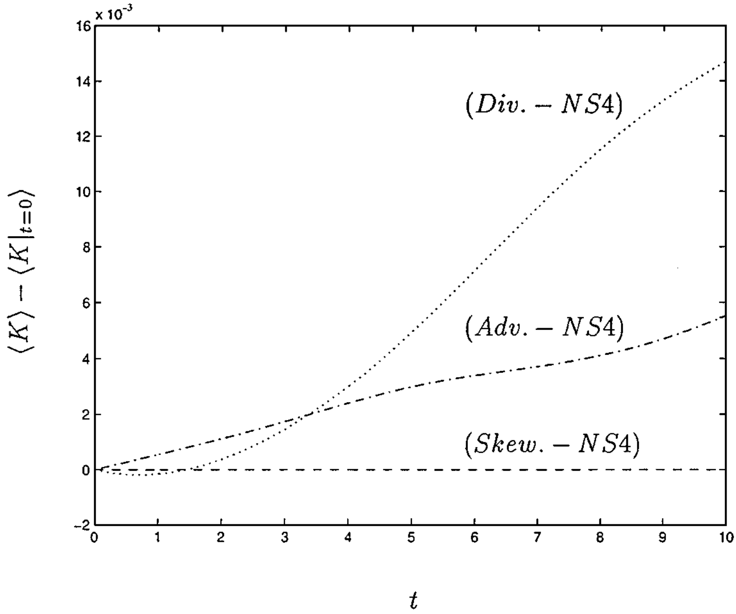


**FIG. 2.** Evolution of the kinetic energy conservation error for (*Div.* − *NS*4) (- - -), (*Adv.* − *NS*4) (—·—), and (*Skew.* − *NS*4) (— — —) convective schemes.

norm for the total kinetic energy,

$$K = \sum_{i=1}^{3} \sum_{x_1} \sum_{x_2} \sum_{x_3} u_i^2(\mathbf{x}) \Delta V(\mathbf{x}), \tag{48}$$

where the sums that appear in Eq. (48) are taken in the respective directions, $\Delta V(\mathbf{x}) \equiv J_2 \Delta V_\xi$,

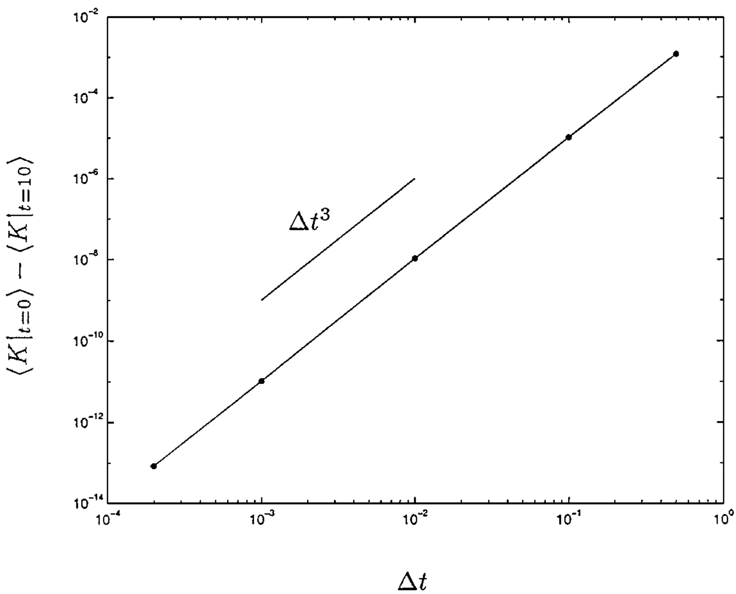


**FIG. 3.** Kinetic energy conservation error at $t = 10$ as a function of time step $\Delta t$ for (*Skew.* − *NS*4) convective scheme.

$J_2$ is the Jacobian of the transformation $x_2 \to \xi_2$, and $\Delta V_\xi = \prod_{k=1}^{3} \Delta_k$ is a constant volume in the computational domain. The energy norm (48) is not conserved for both divergence and advective forms of the convection term. However, an alternative energy norm may be conserved. For that reason further investigation is needed to confirm or deny the existence of such a norm.

The time evolution of the total kinetic energy defined by Eq. (48) is shown in Fig. 2. It can be easily seen that for both divergence and advective forms of the convective term the energy is not conserved. Also it should be noticed that the sign of the conservation energy is not defined since the conservation error is given by the nonlinear term, which can be either positive or negative.

The conservation of the kinetic energy for the skew-symmetric form is confirmed in Fig. 3. Kinetic energy is not conserved exactly since the third order Runge–Kutta time stepping method introduces a slight dissipative error. To demonstrate that the skew-symmetric scheme is conservative, the time step is decreased and the error is compared against the time step. As expected, the time stepping error decreases with the cube of $\Delta t$ (see Fig. 3), and we observe no violation of kinetic energy conservation due to the spatial scheme.

## 6. CONCLUSIONS

The class of high order staggered grid finite difference schemes proposed by Morinishi *et al.* [3] is generalized to non-uniform meshes. The proposed schemes do not simultaneously conserve mass, momentum, and kinetic energy. However, depending on the form of the convective term, conservation of either momentum or energy in addition to mass can be achieved. Furthermore, the non-conservation is weak; it is a function of the commutation error, which is very small for smoothly varying meshes. Certainly, experience has shown that schemes that are fully conservative on uniform meshes perform considerably better on non-uniform meshes when compared to the schemes which are not fully conservative even on uniform meshes. The results presented in this paper are not discouraging at all: the same kind of analysis for the standard generalization to a non-uniform grid of the second order scheme of Harlow and Welch [1] would lead to similar conclusions. Thus, the generalized schemes developed in this paper will enable us to perform numerical simulations with greater accuracy while preserving the conservation properties of the second order scheme of Harlow and Welch.

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